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Short Communication

Gas Phase of Asymmetric Nearest Neighbor Ising Model

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We consider an asymmetric d-dimensional, $d > 1$, Ising model with the pair interaction I in one direction different from the pair interaction J in all other directions. We show that for any inverse temperature β the system is in the gas phase as soon as $|J| < C\beta^{-1}d^{-2}(1-\tanh(\beta\,|I|))$ with $C > 0$ being a small numeric constant.

KEY WORDS: Asymmetric Ising model; cluster expansion.

Asymmetric or anisotropic bounded spin systems are classical models in statistical mechanics. A general idea, suggested originally in [DS2], is to study the system in which the interaction is strong in one direction and weak in other directions in terms of a perturbation of one-dimensional systems. Since the latter do not exhibit phase transitions at any finite temperature it should be possible to find for any fixed value of the strong interaction an orthogonal interaction sufficiently small to ensure the uniqueness of the Gibbs state. Recently this problem has been discussed in various context, see, e.g., [NOZ] and [BK], since it is a basic tool to solve different models both in equilibrium and in non-equilibrium statistical mechanics.

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In this note we consider an asymmetric nearest neighbor Ising model described by the following partition function

$$Z_\Lambda = 2^{-|\Lambda|} \sum_{\{\sigma\}} \exp\left(\beta \sum_{\{x,\, y\} \in \Lambda} J_{xy} \sigma_x \sigma_y\right) \tag{1.1}$$

where $\Lambda \subset \mathbf{Z}^d$ is a cubic box of linear size L containing $|\Lambda| = L^d$ lattice points, the sum $\sum_{\{\sigma\}}$ runs over *spin configurations* $\{\boldsymbol{\sigma}\} \in \{-1, 1\}^{|\Lambda|}$, $\beta > 0$ is the inverse temperature, $\{x, y\}$ denotes unit bonds of Z^d and the coupling J_{xy} is given by

$$J_{xy} = \begin{cases} J & \text{if } \ |x-y| = 1 \ \text{ and } \ x^{(d)} - y^{(d)} = 0 \\ I & \text{if } \ |x-y| = 1 \ \text{ and } \ |x^{(d)} - y^{(d)}| = 1 \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

Here $x = (x^{(1)},..., x^{(d)})$ and $y = (y^{(1)},..., y^{(d)})$ are points of Z^d and $|x-y|$ is the Euclidean distance between them.

Without loss of generality we suppose from now on that $I > J > 0$ and an empty boundary conditions are imposed on Λ. The gas phase is characterized by the fact that $|\Lambda|^{-1} \log Z_\Lambda$ is analytic in β uniformly in the volume Λ. For $d = 2$ an exact solution, see, e.g., [B], shows that for large βI the model given by (1.1) has a unique limit Gibbs state as soon as

$$\beta J < C e^{-2\beta I} \tag{1.3}$$

where C is a numerical constant.

In this note we show by elementary cluster expansion techniques that an estimate of the form (1.3) is also true for $d > 2$ with the constant C depending on d.

Theorem 1. Consider an asymmetric Ising model given by (1.1) and (1.2). Then for any I and β there exists $J_0 = \beta^{-1} C d^{-2} (1 - \tanh(\beta I))$, $C > 0$ such that $|\Lambda|^{-1} \log Z_\Lambda$ can be written as an absolutely convergent series uniformly in $|\Lambda|$ for all $J < J_0$.

Proof. Perform a standard high temperature expansion for the partition function

$$\begin{aligned} Z_\Lambda &= 2^{-|\Lambda|} \sum_{\sigma_\Lambda} \exp\left(\beta \sum_{\{x,\, y\} \in \Lambda} J_{xy} \sigma_x \sigma_y\right) \\ &= 2^{-|\Lambda|} \sum_{\sigma_\Lambda} \prod_{\{x,\, y\} \in \Lambda} \cosh(\beta J_{xy})(1 + \sigma_x \sigma_y \tanh(\beta J_{xy})) \end{aligned}$$

Denote by $|B_w(\Lambda)|$ the number of weak ($=$ orthogonal to $e^{(d)} = (0,..., 0, 1)$) nearest neighbor bonds in Λ and by $|B_s(\Lambda)|$ the number of strong ($=$ parallel to $e^{(d)} = (0,..., 0, 1)$) nearest neighbor bonds in Λ. Then

$$\prod_{\{x,\, y\} \in \Lambda} \cosh(\beta J_{xy}) = \cosh(\beta J)^{|B_w(\Lambda)|} \cosh(\beta I)^{|B_s(\Lambda)|}$$

Moreover

$$\sum_{\sigma_\Lambda} \prod_{\{x,\, y\} \subset \Lambda} (1+\sigma_x\sigma_y \tanh(\beta J_{xy})) = \sum_{\sigma_\Lambda} \prod_{b \in B(\Lambda)} (1+\tilde{\sigma}_b \tanh(\beta J_b))$$

where b denotes a nearest neighbor pair $\{x, y\}$, $\tilde{\sigma}_b = \sigma_x \sigma_y$, $J_b = J_{xy}$ and $B(\Lambda)$ denotes the set of all unit bonds in Λ.

A *connected graph* $g \equiv \{b_1,..., b_n\}$ on Λ is by definition a non empty connected set of distinct pairs $b_i = \{x_i, y_i\} \subset \Lambda$ called *links of the graph.* Given $g \equiv \{b_1,..., b_n\}$ we denote by $|g| = n$ the number of links in g and by $\operatorname{supp} g = \bigcup_{i=1}^n b_i = \bigcup_{i=1}^n (x_i \cup y_i)$ the support of g. Elements of $\operatorname{supp} g$ are called *vertices* of the graph. Given g and $x \in \operatorname{supp} g$ the coordination number d_x of the vertex x is defined as the number of links b in g such that $b \cap \{x\} \neq \varnothing$.

With this definitions we can write

$$2^{-|\Lambda|} \sum_{\sigma_\Lambda} \prod_{b \in B(\Lambda)} (1+\tilde{\sigma}_b \tanh(\beta J_b)) = 1 + \sum_{\substack{\{g_1,..., g_n\} \\ g_i \text{ connected graph} \\ \operatorname{supp} g_i \cap \operatorname{supp} g_j = \varnothing}} \rho(g_1) \cdots \rho(g_n)$$

where

$$\rho(g) = 2^{-|\operatorname{supp} g|} \sum_{\sigma_{\operatorname{supp} g}} \left[\prod_{b \in g} \tilde{\sigma}_b \tanh(\beta J_b) \right]$$

For any connected graph g we have

$$\prod_{b \in g} \tilde{\sigma}_b = \prod_{x \in \operatorname{supp} g} \sigma_x^{d_x}$$

Hence

$$\sum_{\sigma_{\operatorname{supp} g}} \prod_{b \in g} \tilde{\sigma}_b = \begin{cases} 0 & \text{if } d_x \text{ is odd for some } x \in \operatorname{supp} g \\ 2^{|\operatorname{supp} g|} & \text{if } d_x \text{is even for all } x \in \operatorname{supp} g \end{cases} \tag{1.4}$$

and

$$\prod_{b \in g} \tanh(\beta J_b) = \prod_{b \in g} \tanh(\beta\, |J_b|) \tag{1.5}$$

if d_x is even for all $x \in \operatorname{supp} g$.

Define a *contour* γ as a connected graph in Λ such that d_x is even for all $x \in \operatorname{supp} \gamma$. Then

$$2^{-|\Lambda|} \sum_{\sigma_\Lambda} \prod_{b \in B(\Lambda)} (1 + \tilde{\sigma}_b \tanh(\beta J_b)) = 1 + \sum_{\substack{\{\gamma_1, \dots, \gamma_n\} \\ \gamma_i \text{ contour} \\ \operatorname{supp}\gamma_i \cap \operatorname{supp}\gamma_j = \varnothing}} \rho(\gamma_1) \cdots \rho(\gamma_n)$$

where by (1.4) and (1.5) we simply have

$$\rho(\gamma) = \prod_{b \in \gamma} \tanh(\beta\, |J_b|)$$

In terms of contours one has a representation

$$Z_\Lambda = \cosh(\beta J)^{|B_w(\Lambda)|} \cosh(\beta I)^{|B_s(\Lambda)|}\, \Xi_\Lambda$$

where

$$\Xi_\Lambda = 1 + \sum_{\substack{\{\gamma_1, \dots, \gamma_n\} \subset B(\Lambda) \\ \gamma_i \text{ contour} \\ \operatorname{supp}\gamma_i \cap \operatorname{supp}\gamma_j = \varnothing}} \rho(\gamma_1) \cdots \rho(\gamma_n)$$

is the grand canonical partition function of a hard core gas of contours γ with activity $\rho(\gamma)$. The infinite volume pressure of the system is

$$\begin{aligned} p &= \lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \log Z_\Lambda \\ &= \lim_{\Lambda \to \infty} \left[\frac{|B_w(\Lambda)|}{|\Lambda|} \log \cosh(\beta J) + \frac{|B_s(\Lambda)|}{|\Lambda|} \log \cosh(\beta I) \right] + \lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \log \Xi_\Lambda \end{aligned}$$

Clearly $|\Lambda|^{-1}\, |B_w(\Lambda)| \log \cosh(\beta J) + |\Lambda|^{-1}\, |B_s(\Lambda)| \log \cosh(\beta I)$ is analytic for all positive β uniformly in Λ. Therefore to prove the analyticity of p one has simply to check the analyticity of the pressure of the hard core polymer gas. By the standard cluster expansion (see [KP]) the analyticity of $\lim_{\Lambda \to \infty} |\Lambda|^{-1} \log \Xi_\Lambda$ follows from the fact that for some $a > 0$

$$\sup_{x \in \mathbb{Z}^d} \sum_{\gamma:\, x \in \operatorname{supp} \gamma} \rho(\gamma)\, e^{a\, |\operatorname{supp} \gamma|} \leqslant a \tag{1.6}$$

where the sum is taken over all contours whose support contains a given lattice site x. In our case this estimate is a simple exercise in contour summation.

Observe that each contour consists of even number of *elbows*, where an *elbow*, δ, is a connected subgraph of γ consisting of a maximal in γ segment of unit lattice bonds along one direction followed by a maximal in γ segment of unit lattice bonds along an orthogonal direction. More precisely, a connected subgraph δ of $B(\Lambda)$ is an elbow if, for some $1 \leqslant j < k \leqslant d$

$$\delta = \{b_1, \ldots b_s, b'_1, \ldots, b'_\ell\}$$

with $b_i = \{x + (i-1)\, e^{(j)}, x + ie^{(j)}\}$, (for $i = 1, 2, \ldots, s$) and $b'_{i'} = \{x' + (i'-1)\, e^{(k)},\ x' + i'e^{(k)}\}$, (for $i' = 1, 2, \ldots, \ell$) with $x' = x + se^{(j)}$. Given an elbow δ denote by $\{x^\delta, y^\delta\}$ the end-points of the elbow, i.e., the only two vertices of the elbow with coordination number equal to one.

It is now clear that any contour γ can be partitioned (generally not in a unique way) into a union of elbows

$$\gamma = \bigcup_{i=1}^{k} \delta_i$$

Here $(\delta_1, \ldots, \delta_k)$ is a k-tuple of ordered distinct elbows connected in the sense that $y^{\delta_i} = x^{\delta_{i+1}}$ and $x^{\delta_1} = y^{\delta_k}$. The existence of this partition is provided by Euler theorem, which states that for any graph with even coordination numbers at all its vertices it exists a path which passes for every link only once and end in the starting point.

If $(\delta_1, \ldots, \delta_k)$ is a connected k-tuple of ordered distinct elbows and $\gamma = \bigcup_{i=1}^{k} \delta_i$ is the corresponding contour then with

$$\rho(\delta) = \prod_{b \in \delta} \tanh(\beta\, |J_b|)$$

we have

$$\rho(\gamma) = \prod_{i=1}^{k} \rho(\delta_i)$$

and also

$$e^{a\, |\mathrm{supp}\, \gamma|} \leqslant \prod_{i=1}^{k} e^{a\, |\mathrm{supp}\, \delta_i|}$$

Now we can reorganize the sum over all contours whose support contains a fixed point x into a sum over connected $(2k-1)$-tuples of elbows starting at x. Such a $(2k-1)$-tuple can be uniquely completed to a contour by adding an additional elbow which connects the beginning of the first elbow in the $(2k-1)$-tuple with the end of the last elbow in the $(2k-1)$-tuple. Therefore

$$\begin{aligned}\sup_{x \in \mathbb{Z}^d} \sum_{\gamma:\, x \in \operatorname{supp} \gamma} \rho(\gamma)\, e^{a\,|\operatorname{supp} \gamma|} &\leqslant \sup_{\delta} \rho(\delta)\, e^{a\,|\operatorname{supp} \delta|} \sum_{k=1}^{\imath} \sum_{\substack{(\delta_1,\dots,\delta_{2k-1})\\ 0 \in \operatorname{supp} \delta_1}} \prod_{i=1}^{2k-1} \rho(\delta_i)\, e^{a\,|\operatorname{supp} \delta_i|} \\ &\leqslant \sup_{\delta} \rho(\delta)\, e^{a\,|\operatorname{supp} \delta|} \sum_{k=1}^{\imath} \Bigg[\sum_{\delta:\, 0 \in \operatorname{supp} \delta} \rho(\delta)\, e^{a\,|\operatorname{supp} \delta|} \Bigg]^{2k-1}\end{aligned} \tag{1.7}$$

To get an upper bound for the term in square brackets in (1.7) we make the worst hypothesis that the elbow has an arm in the strong coupling direction. Let m be the number of unit bonds in the elbow along the strong coupling direction and n be the number of unit bonds in the elbow along the other direction. Note that the support of the elbow contains $m+n+1$ lattice sites. Then

$$\begin{aligned}&\sum_{\delta:\, 0 \in \operatorname{supp} \delta} \rho(\delta)\, e^{a\,|\operatorname{supp} \delta|} \\ &\qquad < e^a \left(2d \sum_{m=1}^{\infty} \tanh(\beta I)^m\, e^{am} \right) \left(2(d-1) \sum_{n=1}^{\infty} \tanh(\beta J)^n\, e^{an} \right)\end{aligned} \tag{1.8}$$

where $2d$ is the number of possibilities to select a direction of the first leg of the elbow, $2(d-1)$ is a number of possibilities to select a direction of the second leg of the elbow and in the worst situation the first leg goes along the strong coupling direction.

To ensure a convergence of the sum over m we need to choose a in such way that

$$e^a \tanh \beta I < 1 \tag{1.9}$$

For the definiteness we set

$$a = -\tfrac{1}{2} \log \tanh \beta I \tag{1.10}$$

The sum over n in (1.8) also converges because $J < I$. With the choice (1.10) the right hand side of (1.8) can be estimated from above by

$$4d(d-1)\,\frac{\tanh(\beta I)\,e^{2a}}{1-\tanh(\beta I)\,e^{a}}\cdot\frac{\tanh(\beta J)\,e^{a}}{1-\tanh(\beta J)\,e^{a}}$$
$$=\frac{4d(d-1)}{1-[\tanh(\beta I)]^{\frac{1}{2}}}\cdot\frac{\tanh(\beta J)}{[\tanh(\beta I)]^{\frac{1}{2}}-\tanh(\beta J)} \tag{1.11}$$

Also the contribution of a single elbow δ may be bounded by

$$\rho(\delta)\,e^{a\,|\mathrm{supp}\,\delta|}\leqslant[\tanh(\beta J)\,e^{a}]\,[\tanh(\beta I)\,e^{a}]\,e^{a}\leqslant\tanh\beta J(\tanh\beta I)^{-\frac{1}{2}} \tag{1.12}$$

Inserting now (1.11) and (1.12) in (1.7) we complete the proof of the theorem by the following estimate

$$\frac{\tanh\beta J}{(\tanh\beta I)^{\frac{1}{2}}}\sum_{k=1}^{\infty}\left[\frac{4d(d-1)}{1-[\tanh(\beta I)]^{\frac{1}{2}}}\cdot\frac{\tanh(\beta J)}{[\tanh(\beta I)]^{\frac{1}{2}}-\tanh(\beta J)}\right]^{k}$$
$$<-\frac{1}{2}\log\tanh(\beta I) \tag{1.13}$$

which is true for $\beta J < Cd^{-2}(1-\tanh(\beta I))$ and $C > 0$ small enough.

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